Cavitation of a viscous fluid in narrow passages

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The conditions which determine the existence and position of cavitation in the narrow passages of hydrodynamically lubricated bearings have been assumed to be the same as those which produce cavitation bubbles, namely a lowering of pressure below that at which gas separates out of fluid. This assumption enables certain predictions to be made which in some cases are verified, but it does not provide a physical description of the interface between oil and air. Theoretical analysis of the situation seems to be beyond our present capacity, and in none of the experiments so far published has it been possible to measure both the most important relevant data, namely the minimum clearance and the oil flow through it.

A method is described here which enables this to be done. It turns out that two physically different kinds of cavitation can occur. One of these is well described by the existing theory and assumption. Surface tension plays no part in it, and in most text books on hydrodynamic lubrication is not even mentioned. The other kind, which is akin to hydrodynamic separation rather than bubble cavitation, depends essentially on surface tension. Both kinds appear clearly in published photographs taken through transparent bearings, but the experimenters do not seem to have distinguished between them.

The reason why surface tension, which is only able to supply stresses that are exceedingly small compared with the pressure variation in the fluid itself, may have a large effect on the flow can be understood by considering the flow of a viscous fluid in a tube when blown out by air pressure applied at one end. For any given length of fluid the rate of outflow depends almost entirely on the pressure applied, the surface tension force being negligible; but the amount of fluid left in the tube after the air column has reached the end depends essentially on surface tension.

1. Introduction

Cavitation or separation in the oil film of hydrodynamic lubrication has long been recognized as an important factor in the design of bearings and engineers have exercised much ingenuity in trying to allow for it, but rather little effort has been devoted to the study of the phenomenon itself. I think that there are three main reasons for this. The first is that a simple journal bearing is essentially a mechanism by which a rotating shift can be supported against a lateral load so that the geometry of the lubricating fluid depends on the load as well as the peripheral speed and the difference in radii between the shaft and the bearing.

This leads to great complication in the analysis which absorbs much of the attention of the workers in the subject. The second is that it is very difficult to make experiments which can reveal the physical conditions that determine the position of the meniscus which separates the fully lubricated from the separated part of a lubrication film, particularly when the geometry of the solid surface is not predetermined, and in nearly all the recorded experiments it is not. A third reason is that in hydrodynamically lubricated bearings the fluid pressures in the high pressure parts are usually great compared with one atmosphere but the negative pressure in the part (if any) of the bearing where cavitation has occurred is usually very much smaller so that it makes little contribution to the total reaction between bearing surface and shaft. It is sufficient for most practical purposes to assume that the pressure is positive throughout.

The interest, even the practical interest, of cavitation or separation of viscous fluids flowing in narrow passages is not limited to its occurrence in lubrication theory and in other cases, where great pressures do not occur, surface tension, which may justifiably be left out of consideration in most problems in lubrication theory, may be of primary importance.

In designing experiments to determine the physical character of the meniscus separating a viscous lubricant from the outside air it seems essential to reduce the number of variables. The first step is obviously to use predetermined geometry and to limit the motion to two dimensions. The simplest practical way to form a meniscus is to fill a cylindrical tube with viscous fluid and blow it out by air pressure applied to one end. The air forms itself into a column with a round end, which may be called the meniscus. This travels down the tube till it reaches the far end. After the meniscus has passed any point in the tube the fluid which is left behind stays practically at rest, because the viscosity of air is so much less than that of the fluid that the pressure in it is nearly constant along its length. The simplest measurement that can be made is the ratio m of the amount of fluid left behind to the internal volume of the tube, and measurements of this kind have been published (Fairbrother & Stubbs 1935; Taylor 1961; Bretherton 1961). The main result is that, as had been expected on somewhat unsophisticated theoretical grounds, m depends only on the non-dimensional combination $\mu U/T$, where μ is viscosity, U the velocity of the meniscus relative to the wall of the tube and T is surface tension. This may be expressed by the equation

$$m = F_1(\mu U/T). \tag{1}$$

When $\mu U/T$ is small F_1 is small, but as $\mu U/T$ increases F_1 appears to approach an asymptotic limit. In my experiments (Taylor 1961) which extended up to $\mu U/T = 1{\cdot}9$, m had risen to $0{\cdot}56$. In some later experiments by Cox (1962) m nearly reached $0{\cdot}60$ when $\mu U/T$ was in the range 10–$17{\cdot}5$. This point is mentioned here to show that the asymptotic value of m is *not* $0{\cdot}50$ as it would be if the criterion were that the bubble cannot go faster through the tube than the maximum fluid velocity in the Poiseuille flow beyond the bubble which is driven by the air pressure in it. The significance of this fact will appear later.

The other relevant measurement which could be made is the difference in pressure between the fluid on the two sides of the meniscus. It was not possible to do this in my experiments but it is worth while considering the physical meaning of such measurements if they could be made. The flow far ahead of the meniscus is the Poiseuille flow which is associated with a uniform pressure gradient $8a^{-2}\mu V$, a being the radius of the bore of the tube and V the mean velocity through it. To connect the pressure in the air column with the velocity of the meniscus it would be necessary to know how the uniform pressure gradient

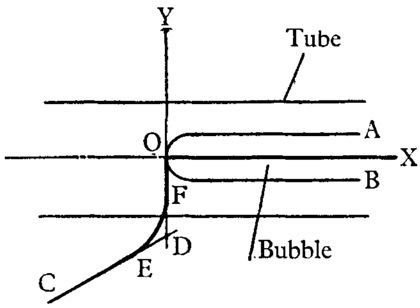


FIGURE 1. Distribution of pressure CEFOX in a tube due to bubble AOB. Pressure excess of that in the bubble is represented by the co-ordinate OY.

in the fluid connects with the practically uniform pressure in the air. The sketch of figure 1 explains what is needed. AOB is the air bubble, OX is the axis of symmetry of the bubble and OY the radial co-ordinate. The distribution of pressure along the axis is represented on an axial plane OXY by the line CEFOX, and the pressure which would have existed at the vertex, O, of the bubble if the uniform gradient which exists in the fluid far away from it had been continued up to the vertex is represented by D. Though it is a difficult matter to calculate the flow near the vertex, all that is necessary to connect the pressure in the air with the flow in the tube is the pressure difference δp represented by OD, and since there is only one non-dimensional variable $\mu U/T$ in the problem, it seems that the pressure change δp must be representable by an expression of the form

$$\delta p = -\frac{\mu U}{a} F_2\left(\frac{\mu U}{T}\right). \tag{2}$$

It will be seen that to calculate how fast fluid would be blown out of a tube by a given pressure both F_1 and F_2 must be known, but since F_2 is likely to be comparable with unity when $\mu U/T$ is large and must tend to $2/(\mu U/T)$ when $\mu U/T$ is small, a knowledge of F_2 is not nearly so important as that of F_1 in most cases unless $\mu U/T$ is small.

The problem presented by a bubble in a capillary tube was considered first because it is the simplest definable and easily realizable case of separation in a viscous fluid when the Reynolds number is so small that only the balance of viscous stresses and surface tension need to be taken into account. The analogous problem where a two-dimensional bubble is forced into fluid contained between two parallel plates cannot be materialized because the meniscus is unstable (Saffman & Taylor 1958), though cases with more complicated geometry, such as the flow when a cylinder rolls on a plane covered with viscous fluid, may involve a stable meniscus. There seems, however, to be no reason why the equilibrium configuration should not be calculated except the great difficulty of doing so, even though it is not stable. This difficulty is increased by the fact that there appears to be no reason, except physical intuition, to suppose that any numerical solution would be unique. This question is given some speculative consideration in the appendix.

2. Flow in a narrow gap between eccentric rotating cylinders

The flow in a long fully lubricated journal bearing is well understood. To avoid geometrical complexities the simplest case will be considered, namely that of a very eccentric bearing, or more definitely, the narrow space or 'nip' where two cylinders or a cylinder and a plane nearly come into contact, and the equations for the flow in that region will be reproduced in order to develop a method for finding where the meniscus could be located. The physical properties of the meniscus will be assumed to be analogous to those of the bubble in a capillary tube, namely that at the meniscus

$$m = F_1(\mu U/T) \tag{3}$$

and

$$-\delta p = \frac{\mu U}{h} F_2(\mu U/T). \tag{4}$$

Here m is the ratio of the amount of fluid flowing at any section to the amount which would flow if the pressure gradient there were zero, and h is the distance between the surfaces.

The use of (3) and (4) involves the assumption, which experiments to be described later seem to confirm, that the small deviation from exact parallelism of the two surfaces will not appreciably affect m or δp. The convergence or divergence of the passage has a great effect on the stability of the meniscus (Pearson 1960; Pitts & Greiller 1961) and on the distribution of pressure in the passage, and in that way will have a predominating effect on the position at which the meniscus will establish itself, but this is no reason for rejecting the relations (3) and (4) which are concerned only with local conditions close to a two-dimensional meniscus.

Two cases may be considered: (*a*) both surfaces are moving with velocity U, so that $m = q/Uh$ where q is the volume flowing past the nip per unit length; (*b*) only one surface is moving, as in the case of a bearing, so that $m = 2q/Uh$.

If x is the distance along the nip measured from the narrowest point where $h = h_0$, the variation of h with x can be expressed approximately by the equation

$$h = h_0 + x^2/2R. \tag{5}$$

Here

$$R = (R_1^{-1} - R_2^{-1})^{-1}, \tag{6}$$

where R_1 and R_2 are the radii of the cylinders, R_2 being the larger and R_1 and R_2 are taken as both positive if the centre of the smaller cylinder is inside the larger cylinder. When the case considered is that of a cylinder rolling on a plane, R is evidently the radius of that cylinder.

It is convenient to use instead of x a co-ordinate θ defined by

$$\tan\theta = x(2Rh_0)^{-\frac{1}{2}}. \tag{7}$$

The equation for the distribution of pressure according to the Reynolds approximation then takes the non-dimensional form

$$\frac{dp'}{d\theta} = \cos^2\theta - \lambda\cos^4\theta. \tag{8}$$

This equation applies to both cases (a) and (b), but in case (a) the pressure p is related to p' by

$$p = p'\{12\mu U(2R)^{\frac{1}{2}} h_0^{-\frac{3}{2}}\}, \tag{9a}$$

and in case (b)

$$p = \tfrac{1}{2}p'\{12\mu U(2R)^{\frac{1}{2}} h_0^{-\frac{3}{2}}\}. \tag{9b}$$

In both cases (a) and (b) λ is the ratio of the total flow through the nip to the flow if there had been no pressure gradient at $\theta = 0$, so that in both cases (a) and (b)

$$\lambda h_0 = mh. \tag{10}$$

The solution of (8) is

$$p' = \tfrac{1}{2}\theta + \tfrac{1}{4}\pi + \tfrac{1}{4}\sin 2\theta - \lambda(\tfrac{3}{8}\theta + \tfrac{3}{16}\pi + \tfrac{1}{4}\sin 2\theta + \tfrac{1}{32}\sin 4\theta), \tag{11}$$

where the constant of integration has been chosen so that $p' = 0$ when $\theta = -\frac{1}{2}\pi$, i.e. the pressure is atmospheric far from the nip in the upstream direction, or in other words it is flooded upstream.

When $\lambda = \frac{4}{3}$, $p = 0$ at $\theta = +\frac{1}{2}\pi$ as well as at $\theta = -\frac{1}{2}\pi$ and the distribution of p' is antisymmetrical, being positive when $\theta < 0$ and negative when $\theta > 0$. The distribution of p' in this case is shown in figure 2.

3. Position of meniscus

To find the position of the meniscus for given values of $\mu U/T$ and h_0/R a diagram like figure 2 may be used. Lines showing the variation of p' with θ for constant λ can be plotted, but it is not necessary to cover the whole field because, as has been noted many times in the literature of lubrication theory, the flow can only divide at a plane where dp/dx is positive. Figure 3, which covers a part of the field where a meniscus could occur, has therefore been prepared using the expression (11) and calculating p' for given values of θ and λ.

The diagram of figure 3 is particularly suitable for finding the locus of the points where condition (3) can be satisfied both in case (a) and in case (b). The definition of m as the ratio of the amount of fluid passing through the nip to the amount which would flow if there were no pressure gradient leads to

$$m = \lambda h_0/h = \lambda\cos^2\theta. \tag{12}$$

Thus lines of constant m can be drawn and, according to the condition (3), these will be lines of constant $\mu U/T$. Such lines are shown in figure 3 for values of m from 0·03 to 0·70 and for $m = 1$. $m = 1$ is evidently the line which represents the points where the flow could divide without changing velocity, that is it is the line $dp'/d\theta = 0$ or, using equation (12), $\cos^2\theta = \lambda^{-1}$.

To fix the point in figure 3 which represents the position of the meniscus it is necessary to know h_0 as well as $\mu U/T$. Assuming that we have measured $F_2(\mu U/T)$ the corresponding values of p' using (4) and (9) are

$$-p' = \frac{1}{12}\left(\frac{h_0}{2R}\right)^{\frac{1}{2}} F_2\left(\frac{\mu U}{T}\right)\cos^2\theta \quad \text{in case } (a), \tag{13a}$$

or

$$-p' = \frac{1}{6}\left(\frac{h_0}{2R}\right)^{\frac{1}{2}} F_2\left(\frac{\mu U}{T}\right)\cos^2\theta \quad \text{in case } (b). \tag{13b}$$

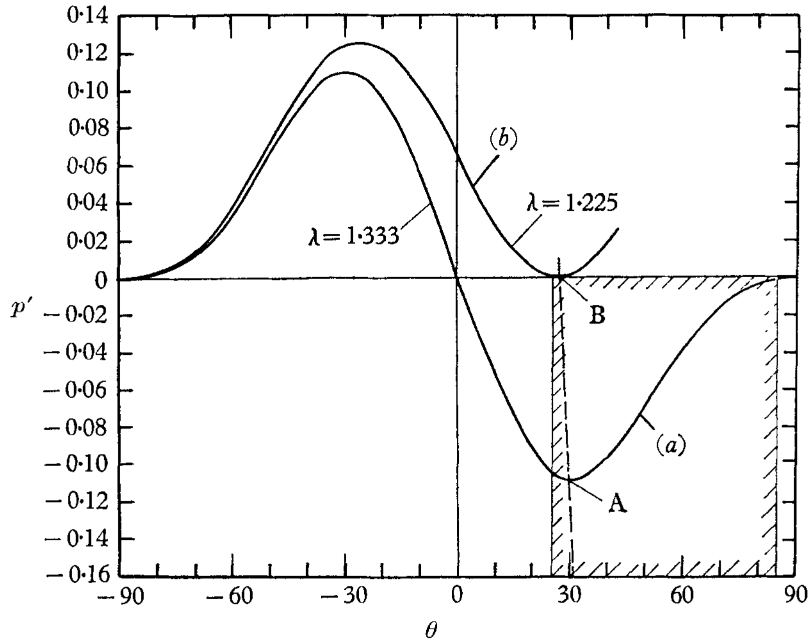


FIGURE 2. Distribution of pressure (*a*) when completely flooded and (*b*) when the Swift–Stieber condition is satisfied at B.

If F_2 were measured experimentally or calculated as a function of $\mu U/T$, the expressions (13) would be used to superpose lines of constant h_0/R on figure 3, and the intersections of these lines with those of constant m and therefore constant $\mu U/T$ would determine the points at which the meniscus would lie for any given value of $\mu U/T$ and h_0/R. It will be noticed that when h_0/R is very small, as it is in most bearings, p' is also small so that unless $F_2(\mu U/T)$ is very large the position of the meniscus is determined simply by the point where the axis $p' = 0$ cuts the appropriate line of constant $\mu U/T$. In other words it is only necessary to know $F_1(\mu U/T)$ in such cases.

To discuss the stability of the meniscus a knowledge of $F_2(\mu U/T)$ is essential. In discussing the origin of the streaks formed when a viscous fluid is spread on a flat sheet by means of a roller, Pearson (1960) assumed that the pressure difference between the two sides of a meniscus was $2T/h$, where h is the distance between the solid surfaces at the position of the meniscus. This is equivalent to assuming that the viscous stress in the neighbourhood of the meniscus which is of order $\mu U/h$ is small compared with $2T/h$, or in other words, that $\mu U/T$ is small. $F_2(\mu U/T)$ is then large, for by the definition in (4), $F_2(\mu U/T) = 2T/\mu U$.

The investigation can only be extended to cases where $\mu U/T$ is not small either by making an arbitrary assumption that FD in figure 1 is small compared with OF or by including $F_2(\mu U/T)$ as an arbitrary constant, an assumption which is legitimate when U is constant as it is in cases (a) and (b).

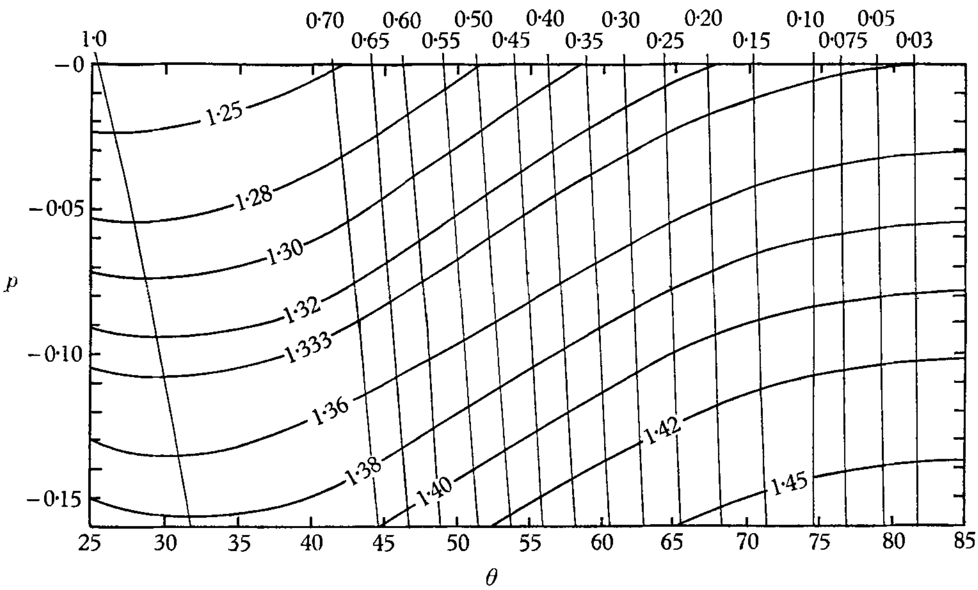


FIGURE 3. Contours of p' for constant values of λ (1·25 to 1·45), and of m (0·03 to 0·70 and 1·0).

4. Apparatus for measurements of $F_1(\mu U/T)$

The main difficulty in measuring m is to obtain two-dimensional flow because when the two surfaces are parallel the meniscus is unstable (Saffman & Taylor 1958). It is true that the meniscus may be stable in the diverging region downstream of the nip, but the conditions which lead to such stability are complicated and not very well understood (Pearson 1960). Another difficulty is that of measuring m in a journal bearing. To collect and measure q, the volume which passes the nip per second would be difficult. In designing apparatus for determining m as a function of $\mu U/T$ it seemed best to attempt to control m and measure U. If the meniscus had been stable this would perhaps have involved varying U till the meniscus was brought to rest, but the instability prevented the direct use of this method. It was found, however, that the configuration illustrated in figure 4 stabilized the meniscus at slow speeds.

A Perspex block was cut accurately rectangular and a trough of uniform depth 0·05 cm and length 12·4 cm was cut in its lower face. This trough did not extend to the front end of the block. The apparatus could slide on a piece of plate glass, selected optically for flatness by Messrs Pilkington, and as it moved it deposited a sheet of fluid 0·025 cm thick on the glass. The fluid was supplied at a vertical chamber cut in the Perspex as indicated at the section AA in figure 4. The object of the long shallow trough hereafter called the regulating chamber was merely to regulate the supply of fluid to an observation chamber between the glass and a second short adjustable block of Perspex which is shown in the section CC. This block was 4·5 cm long. If h_0 is the depth of the regulating trough and h_1 the

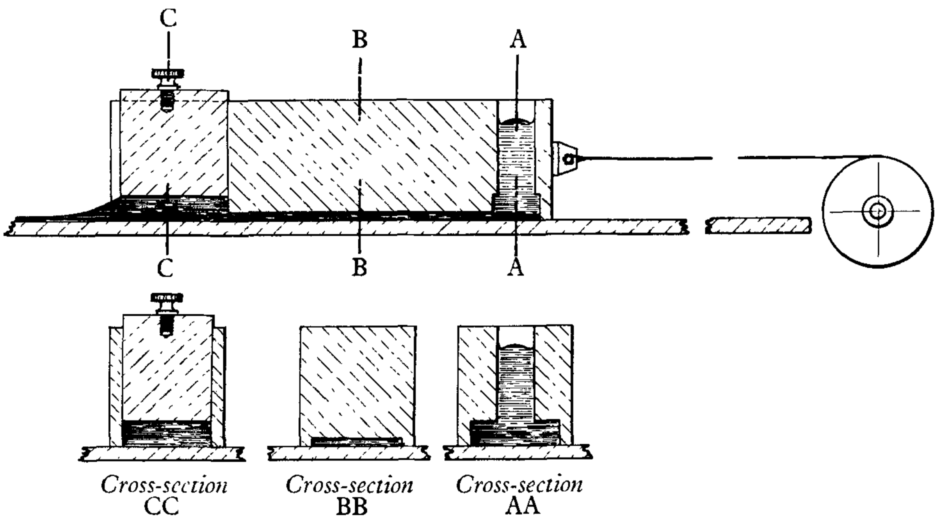


FIGURE 4. Perspex block apparatus.

height of the channel beneath the movable block, the flow in this chamber can be defined by the ratio

$$m = \frac{\text{actual flow in observation chamber}}{\text{flow which would exist if there were no pressure gradient there}}.$$

If the regulating trough were very long the flow in it would simply be $\frac{1}{2}Uh_0$, where U is the velocity of the block over the glass. The actual flow in the chamber at C is therefore $\frac{1}{2}Uh_0$, but if there were no pressure gradient in the observation chamber the flow would be $\frac{1}{2}Uh_1$ so that

$$m = h_0/h_1. \tag{14}$$

Owing to the fact that the length of the observation chamber C was small compared with that of the regulating chamber it was first considered justifiable to take m as h_0/h_1. Later a correction (see equation (18)) was applied which took account of the fact that the length of the observation chamber was comparable with that of the regulating chamber, but did not allow for changes in pressure between the two sides of the meniscus due to surface tension.

The experiment consisted in towing the block in a straight line over the glass at gradually increasing speeds. The meniscus was always hollow, but at slow speeds it sloped away from the upper block as indicated in figure 4. As the speed increased the meniscus became hollower till finally it became apparently horizontal and tangential to the observation chamber at the top. At that stage its horizontal sections were straight lines. A small increase in speed then made the meniscus begin to curve slightly in horizontal sections so that it was no longer two-dimensional. As the speed increased the middle of the meniscus withdrew from the middle of the trailing edge of the observation chamber and the curvature of horizontal sections increased till an unstable condition was reached at which it suddenly began to move forward without any further increase in the speed of the block. The meniscus frequently divided but it always ran forward as far as the rear end of the regulating channel. The value of U which was measured was that at which curvature first appeared in the horizontal sections, because that was the point at which the meniscus was two-dimensional, i.e. cylindrical, and tangential to the top of the observation chamber. The values of U so found were multiplied by μ/T and are plotted against h_0/h in figure 6.

5. New apparatus

The curvature in horizontal sections of the meniscus was no doubt due to the fact that the channel was not very wide compared with its depth. A wider block would certainly have reduced the gap between the speed at which the horizontal curvature began and the speed at which the meniscus ran forward unstably to form air fingers. Even so the block had a considerable disadvantage because the glass plate on which it moved was only 30 in. long, and with the larger values of h_0/h_1 the flow pattern had not settled down to its final steady state by the time the block had traversed 30 in.

To overcome these difficulties a new apparatus was designed in which a cylinder could rotate continuously in a trough of fluid and a fixed concentric arc with clearance h_0 was used to regulate the flow into an observation chamber. This apparatus is shown in figure 5. A cylinder B of radius 7·60 cm and length 38·0 cm could rotate in ball bearings held in a strong steel frame. A regulating Perspex block A whose lower surface was a cylinder of radius 7·65 cm was fixed concentric with the cylinder so that the resulting space between them was 0·05 cm thick. The angle covered by this block was 78°,* so that the length of the regulating space was 10·4 cm. The width in the direction of the generators of the cylinder was 22·4 cm. The accuracy of the setting was ensured by laying flexible spacers 0·050 cm thick on the cylinder and then bringing the regulating block onto them. In this position the block was fixed to a strong frame by suitable adjusting screws. The fluid was contained in a trough C (figure 5) which was filled up to the level of the top of the cylinder. Leakage at the point where the $\frac{1}{2}$ in. steel axle of the cylinder passes through the ends of the trough C was prevented by means of O-rings set in the steel frame. The trough which was made of tinned steel sheet was slightly flexible so that as the trough filled a small leak began to develop as the weight of fluid increased during filling. This was cured by adjusting supports between the strong frame and the trough.

The observation chamber in which the meniscus was observed was the space under a Perspex block D (figure 5) which was fixed by bolts E passing through slots in D which allowed for adjustment of the distance h_1 of D from the cylinder.

* When figure 5 was drawn the dimensions of the regulating block A had been mislaid so that the position of its lower corner is incorrectly shown. The angle 78° was found on dismantling the apparatus.

The bottom of the block D was a part of a cylinder of radius 7·65 cm. To set up the apparatus flexible strips of the required thickness were made and placed on the cylinder. The block D was then laid so that its rear face was in contact with the forward face of the regulating block (which was in an axial plane), and its forward bottom corner F was in contact with the flexible spacers. Since the spacers were lying on the cylinder and the block covered an angle of 36° the thickness of the observation chamber was slightly greater at the rear than at the forward end

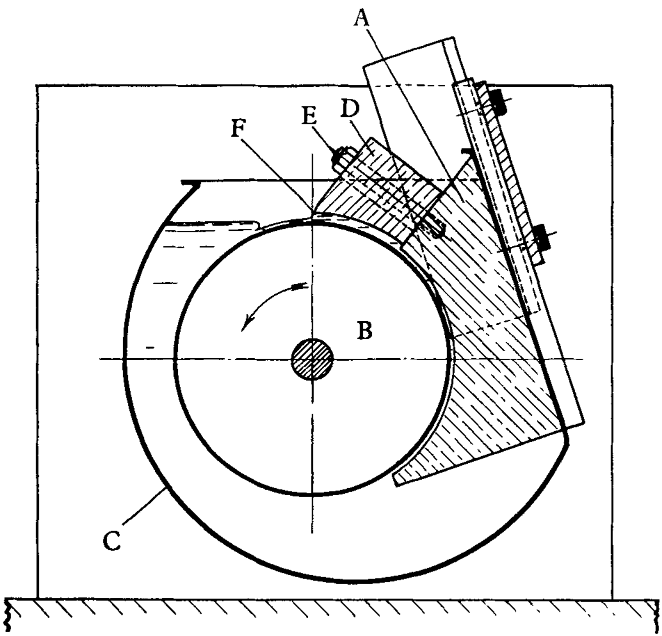


FIGURE 5. Cylindrical apparatus.

by an amount $(h_1 - h_0)$ (sec 36° − 1). The cylinder was driven through a train of continuously varying and fixed reduction gears by a constant speed motor so that the peripheral velocity U of the cylinder could be slowly increased till the meniscus withdrew from the forward end F of the observation chamber. As with the Perspex slider shown in figure 4, the cavitating air-fingers always ran forward to the rear end of the regulation chamber. The speed U at which this occurred could be determined with good repeatability, but to ensure that the flow was two-dimensional it was necessary to fit end-plates lined with sorbo-rubber or felt which could be fitted to the end of regulating block A and to D to prevent air or fluid from being sucked laterally into the rear end of the observation chamber where the lowest pressures occurred. Even this precaution had to be supplemented by guide vanes outside the end-plates to ensure that the outside of the felt or sorbo rubber lining was flooded with the fluid, for it was found that if any air got through, bubbles in the observation chamber destroyed the two-dimensional character of the flow and upset the conditions leading to the instability of the meniscus.

The fluids used were pure glycerine and glycerine diluted with 5 % water. The viscosity of the fluid used was measured at a range of temperatures and the temperature was measured before and after each experiment. Gylcerine does not wet Perspex completely and a thin sheet of glycerine on the surface develops dry areas after a time. This fact does not seem to affect the critical value of U, for in the Perspex block experiments both wetting oils and glycerine were used and no difference was found between the results obtained. Oil was not used in the cylinder apparatus because of the difficulty of cleaning it. The surface tension T was taken as 63 dyn/cm throughout because the variation with temperature

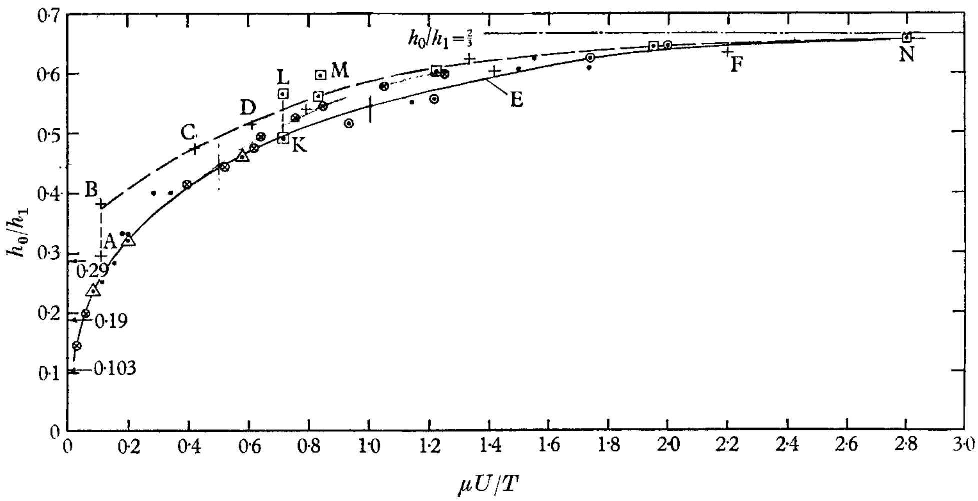


FIGURE 6. Measurements of $\mu U/T$ at which flow ceased to be two-dimensional for fixed values of h_0/h_1. Parallel observation chamber: △ $h_0 = 0{\cdot}05$ cm; • $h_0 = 0{\cdot}10$ cm; ⊙ $h_0 = 0{\cdot}15$ cm. Rectangular block: ⊙ $h_0 = 0{\cdot}05$ cm. Expanding observation chamber: + angle = 2·8°; ⊡ angle = 1·3°. Limiting values for $h_0 = 0{\cdot}05$, 0·10, 0·15 cm.

is small. Some uncorrected results obtained with the regulating chamber of uniform depth $h_0 = 0{\cdot}05$ cm are marked in figure 6 by the symbol △. It will be seen that these are in fairly good agreement with those obtained with the flat Perspex block, at any rate for values of $\mu U/T$ up to 0·6. At higher values of $\mu U/T$ the limitations of the flat block apparatus were making the results unreliable.

6. Experiments with larger regulating channel

As the speed rose the difficulty of excluding bubbles from the chamber increased and the highest values of $\mu U/T$ obtainable was 0·6. To increase $\mu U/T$ without increasing the suction at the rear end of the observation chamber it was necessary to increase h_0. To make a regulating space uniform with thickness 0·1 cm the cylindrical surface of the block A (figure 5) should have been re-machined to a radius 7·70 and in fact this has now been done, but the experiments have not yet been repeated with the re-machined block. On the other hand, the flow regulating effect of a long narrow space upstream of the observation chamber does not depend on its having a uniform thickness, but a correction must be applied when it does not.

7. Corrections

If h is known as a function of x the distance along the regulating block, the equivalent thickness of the block of the same length but uniform thickness which would deliver the same volumetric rate of flow can be found by integrating Reynolds's equation. If q is the volumetric rate of flow per cm of the meniscus, Reynolds's approximation is

$$\frac{1}{12\mu}\frac{dp}{dx} = \frac{U}{h^2} - \frac{q}{h^3} \quad \text{in case } (a), \tag{15a}$$

and

$$\frac{1}{12\mu}\frac{dp}{dx} = \frac{U}{2h^2} - \frac{q}{h^3} \quad \text{in case } (b). \tag{15b}$$

When h is uniform and there is no pressure gradient $h = 2q/U$ in case (b), and if h is variable but there is no difference in pressure at the two ends of the channel

$$q\int\frac{dx}{h^3} = \tfrac{1}{2}U\int\frac{dx}{h^2} \tag{16}$$

the integrals extending along the block. Hence

$$m = \frac{2q}{h_1 U} = \int h^{-2}dx \Big/ h_1 \int h^{-3}dx, \tag{17}$$

the integrals extending the total length of the regulating and observation chambers, but it is convenient to express (17) non-dimensionally in the form

$$m = \frac{h_0}{h_1}\int\left(\frac{h_0}{h}\right)^2 dx \Big/ \int\left(\frac{h_0}{h}\right)^3 dx, \tag{18}$$

where h_0 is to be taken as some easily measurable quantity.

To set the regulating block so that it formed a channel which was wider than 0·05 cm, flexible spacers 0·10 cm thick were laid on the cylinder and the Perspex block of radius 7·65 cm was brought down onto them and fixed rigidly. The depth of the channel at all points of the regulating space has first to be found. At its two ends it is h_0 ($h_0 = 0{\cdot}1$ cm in the present case). If the radius of the lower surface of the regulating block is $R+\delta$, R being that of the cylinder ($\delta = 0{\cdot}05$ cm in the present case), the depth h of the regulating space can be found by using the consideration that the method of setting up the regulating block ensures that h is symmetrical about its mid-point. If ϕ is the angular co-ordinate and the mid-point of the regulating space is $\phi = 0$, it is found that

$$h = h_0\{\alpha + (1-\alpha)\sec\phi_0 \cos\phi\}, \tag{19}$$

where $\alpha = \delta/h_0$ and $2\phi_0$ is the angle subtended by the regulating block (in this case 78°). To apply equation (19) it is sufficiently accurate to take the depth in the short observation chamber as uniform and equal to h_1 which is the measured depth of the channel at the forward end F (figure 5). The contributions of the observation chamber to the integrals in (18) are then $(h_0/h_1)^2\phi_2$ and $(h_0/h_1)^3\phi_2$ to the numerator and denominator, respectively, where ϕ_2 is the angle covered by the observation chamber (in the present case $\phi_2 = 36°$). The expression for m is therefore

$$m = \frac{h_0}{h_1} \frac{\left(\frac{h_0}{h_1}\right)^2 \phi_2 + \int_{-\phi_0}^{+\phi_0} \{\alpha + (1-\alpha)\sec\phi_0 \cos\phi\}^{-2} d\phi}{\left(\frac{h_0}{h_1}\right)^3 \phi_2 + \int_{-\phi_0}^{+\phi_0} \{\alpha + (1-\alpha)\sec\phi_0 \cos\phi\}^{-3} d\phi}. \tag{20}$$

For the case when $\delta = h_0 = 0{\cdot}05$ cm, $\alpha = 1$ and equation (20) reduces to

$$m\frac{h_1}{h_0} = K_{0\cdot05} = \frac{(h_0/h_1)^2 + \frac{78}{36}}{(h_0/h_1)^3 + \frac{78}{36}}, \tag{21}$$

where $K_{0\cdot05}$ is a correcting factor to be applied to the approximate equation (14). The correcting factor K_{block} for the Perspex block which had a regulating channel 12·4 cm long and an observation chamber 4·5 cm long is the same as (21) except that the fraction 78/36 is replaced by 12·4/4·5. The calculated values of $K_{0\cdot1}$, $K_{0\cdot05}$ and K_{block} are given in table 1. The value of $K_{0\cdot1}$ was calculated numerically using (20).

h_0/h_1	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
$K_{0\cdot1}$	1·095	1·108	1·125	1·142	1·153	1·156	1·147	1·122
$K_{0\cdot05}$	1·004	1·015	1·029	1·054	1·055	1·060	1·058	1·048
K_{block}	1·006	1·010	1·020	1·035	1·046	1·047	1·050	1·040

TABLE 1. Correction factors to be applied to measured values of h_0/h_1.

8. Effect of gravity

The neglect of the effect of gravity at the meniscus produces little error except at very low speeds. There is a limiting value of h_1 above which the meniscus will leave the leading edge of the observation chamber no matter how small U may be. Since the thickness of the fluid sheet which is carried away is $\frac{1}{2}h_0$ the height of the bottom of the observation chamber above the top of the sheet is $h_1 - \frac{1}{2}h_0$ and the hydrostatic force, which must be balanced by the surface tension force $2T$, is $\frac{1}{2}\rho g(h_1 - \frac{1}{2}h_0)^2$, where ρ is the fluid density. Thus the minimum value of m when $U = 0$ is $h_0/\{\frac{1}{2}h_0 + (4T/\rho g)\}^{\frac{1}{2}}$. For glycerine $(4T/\rho g)^{\frac{1}{2}} = 0{\cdot}46$ cm so that the value of m at $U = 0$ is $h_0/(\frac{1}{2}h_0 + 0{\cdot}46)$. When h_0 is 0·05 cm this is $m = 0{\cdot}103$, for $h_0 = 0{\cdot}10$ cm it is $m = 0{\cdot}19$ and for $h_0 = 0{\cdot}15$ cm it is $m = 0{\cdot}29$. These limits are marked on figure 6. At first sight one might be inclined to think that since the value of m at $U = 0$ is comparable with that at finite values of $\mu U/T$ large errors might arise owing to the effect of gravity. This, however, seems to me unlikely because as soon as U is finite a comparatively large pressure defect at the meniscus can be built up by a very small change in the flow through the regulating channel, and all that gravity does is to change very slightly the rate of flow which is necessary to set up the conditions at the meniscus which lead to its retreat into the observation chamber. The values of m deduced by applying the correcting factors of table 1 are shown in figure 7.

The principal generalizations that these experiments suggest are as follows.

(1) The value of m at which the meniscus begins to retreat into the channel is a function of $\mu U/T$ only.

(2) As $\mu U/T$ increases m appears to approach an asymptotic value which is certainly above 2/3, the value it has when the flow close to the fixed surface begins to reverse its direction at points in the chamber where the effect of the meniscus in deflecting the streamlines is negligible and the Reynolds approximation holds. This criterion was proposed by Hopkins (1957) and used for experiments on flow of type (*a*) where it predicts $m = \frac{1}{3}$.

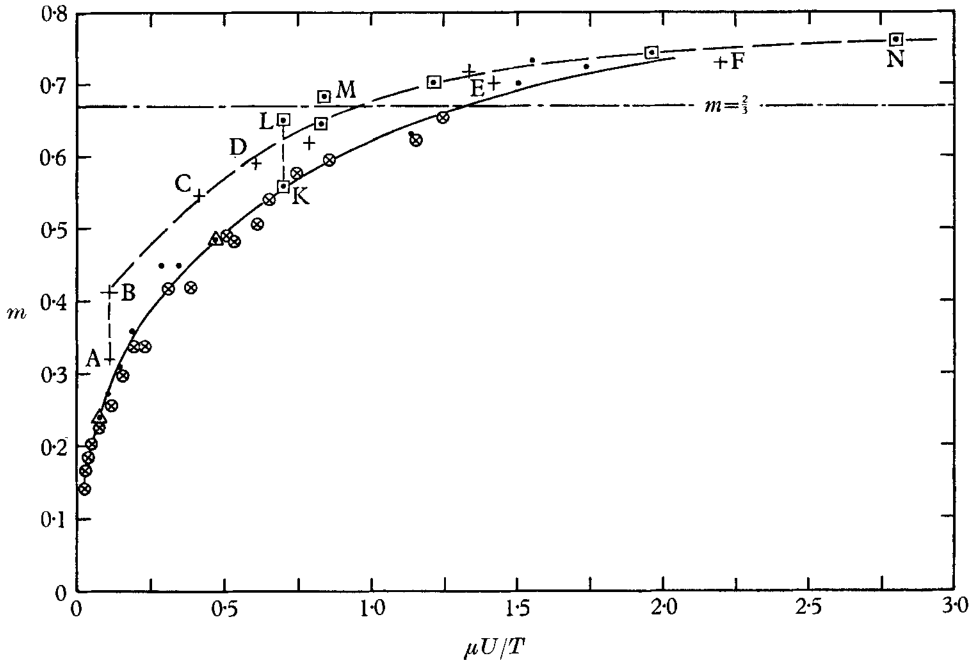


FIGURE 7. Corrected values of m. Parallel observation chamber: • $h_0 = 0{\cdot}10$ cm; △ $h_0 = 0{\cdot}05$ cm. Expanding observation chamber: + angle = 2·8°; □ angle + 1·3°. Rectangular block: ⊙ $h_0 = 0{\cdot}05$ cm.

(3) At small values of $\mu U/T$ the curve gives the impression of being parabolic. A similar approximation has been noted in the case of a bubble in a capillary tube (Fairbrother & Stubbs 1935; Taylor 1961), though it has been shown (Bretherton 1961) that the approximation ceases to be valid when m is less than about 10^{-3}. A rough approximation for the parabolic range of figure 7 is

$$m = 0{\cdot}85(\mu U/T)^{\frac{1}{2}}. \tag{22}$$

Fairbrother & Stubbs's empirical formula was $m = 1{\cdot}0(\mu U/T)^{\frac{1}{2}}$ but there is no reason to expect exact agreement between the two formulae.

9. Experiments with diverging observation chamber

The fact that the meniscus does not seem to be able to travel back when m is greater than some number which is in the neighbourhood of 2/3 suggests that if the observation channel were made to diverge instead of being parallel so that m could vary through the observation chamber from some small number to 1·0, the air cavity would not be able to reach the regulating block but would stop at some intermediate point. To make the observation chamber diverge some thin

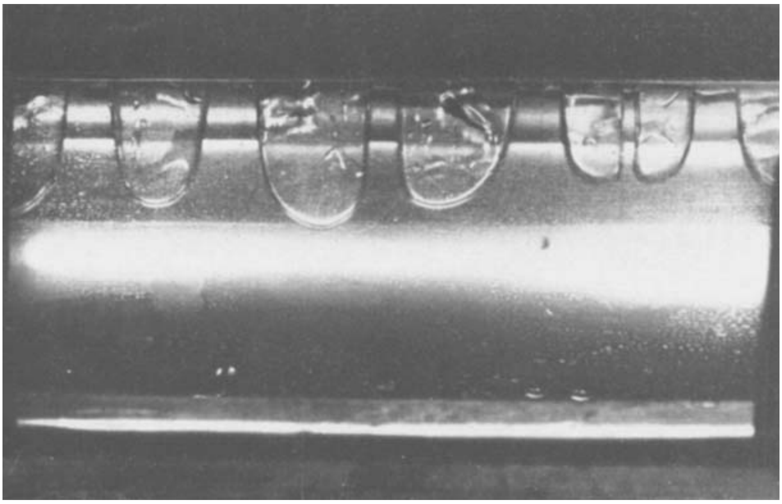


FIGURE 8. Air fingers corresponding with $\mu U/T = 0{\cdot}11$ and divergence $1{\cdot}3°$, $h_0 = 0{\cdot}05$.

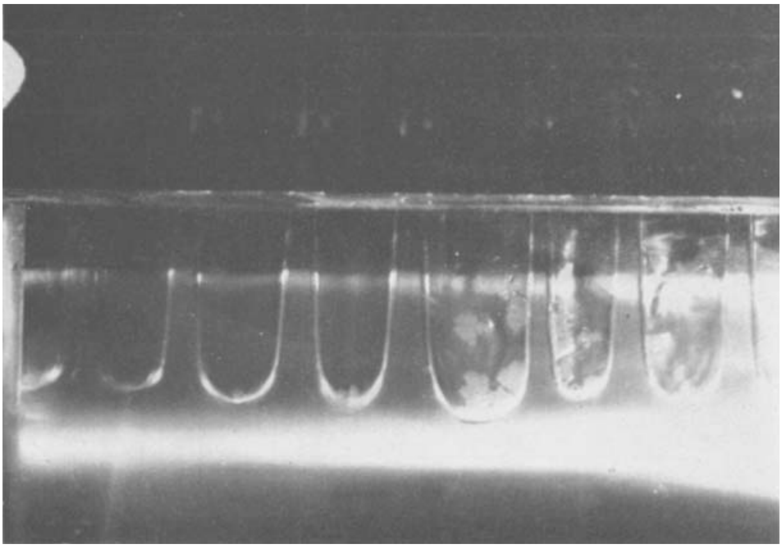


FIGURE 9. $\mu U/T = 1{\cdot}3$, $h_0 = 0{\cdot}05$, divergence $2{\cdot}8°$.

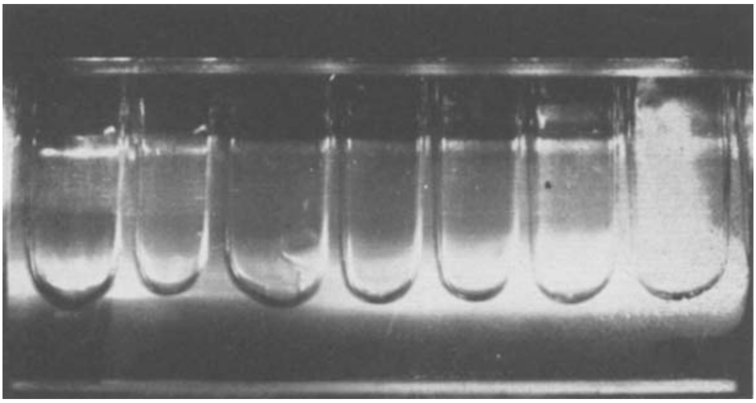


FIGURE 10. $\mu U/T = 2{\cdot}2$, $h_0 = 0{\cdot}05$, divergence $2{\cdot}8°$.

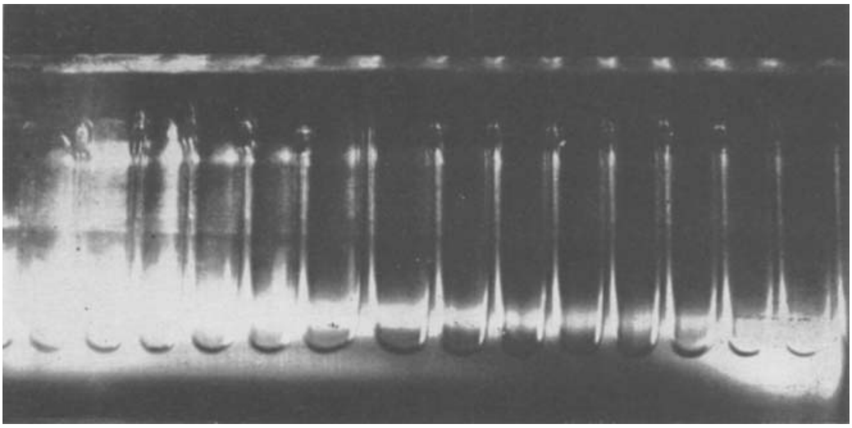
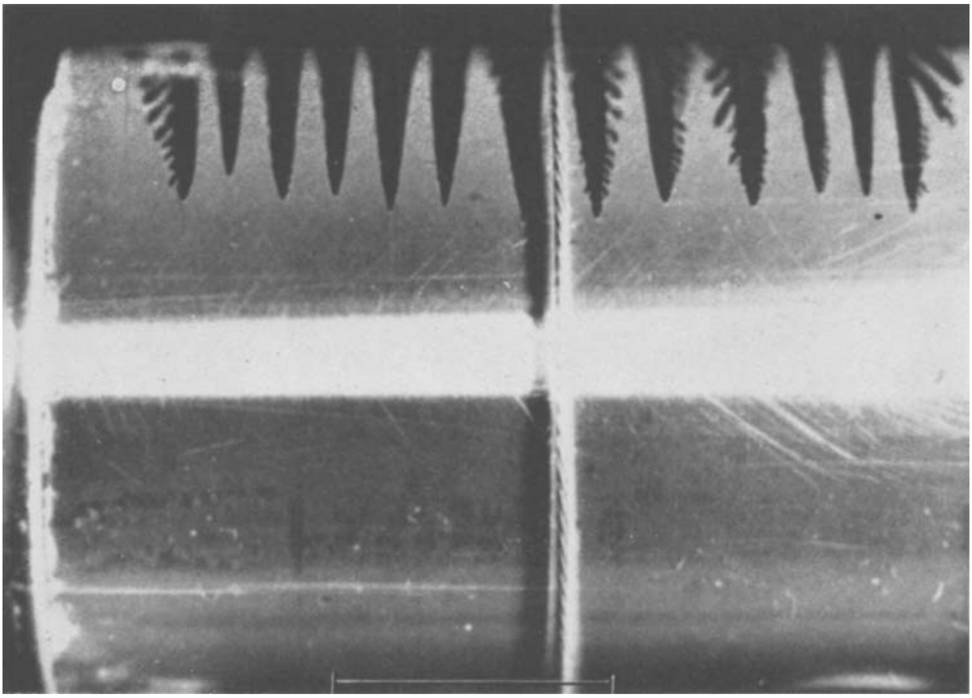
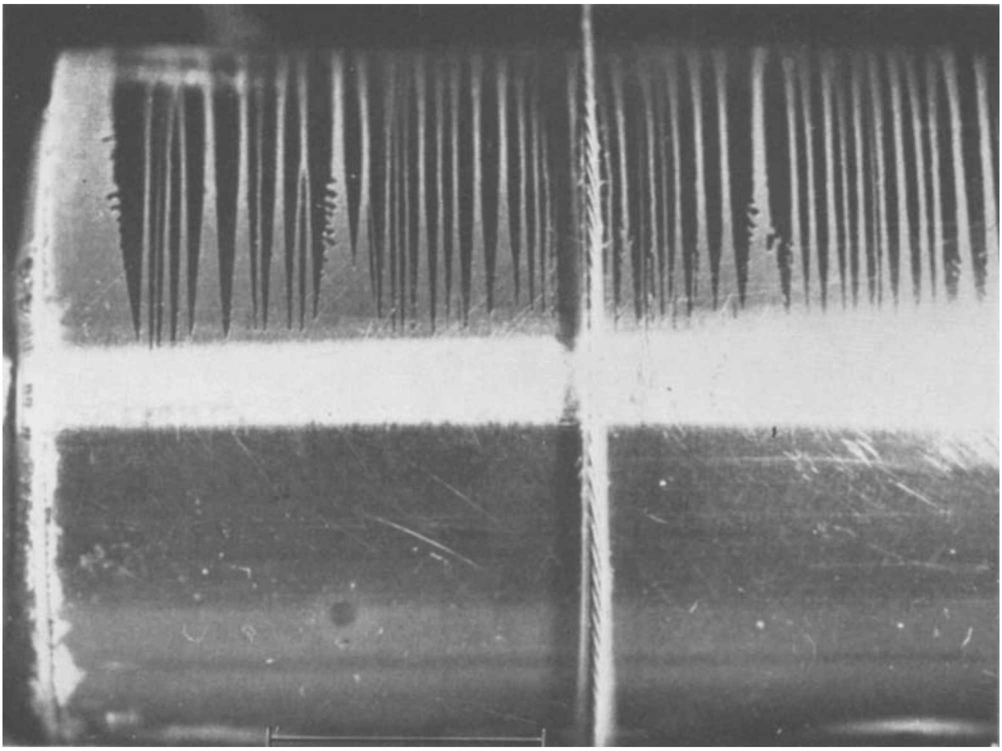


FIGURE 11. $\mu U/T = 4{\cdot}0$, $h_0 = 0{\cdot}05$, divergence $2{\cdot}8°$.



1 cm

FIGURE 12. Cavitation between shaft and transparent bush at $\mu U/T \sim 0{\cdot}12$, $h_0 \sim 1{\cdot}0 \times 10^{-3}$ cm.



1 cm

FIGURE 13. Cavitation between shaft and transparent bush at $\mu U/T \sim 0{\cdot}03$, $h_0 \sim 0{\cdot}3 \times 10^{-3}$ cm.

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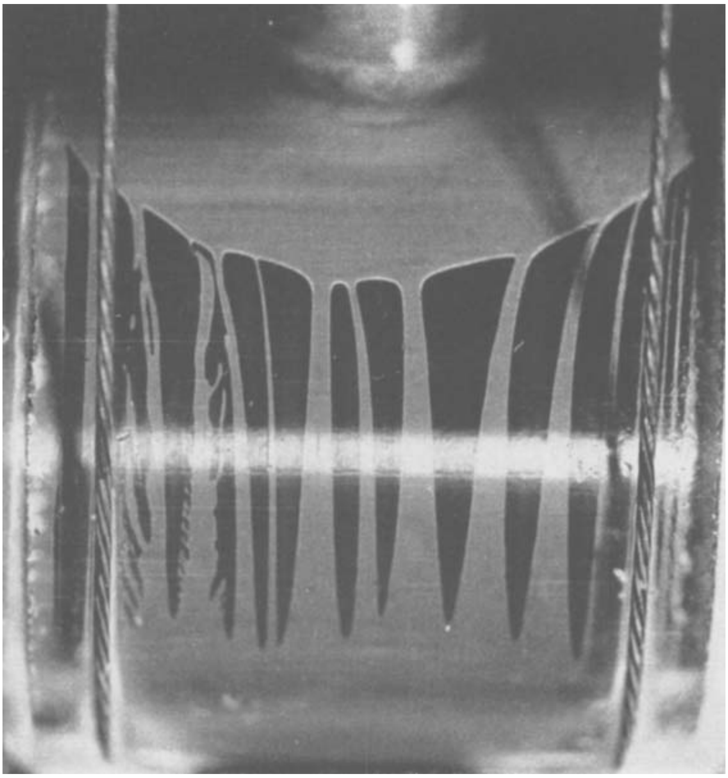

FIGURE 14. Photograph showing both the formation of air fingers and the reformation of the oil film.

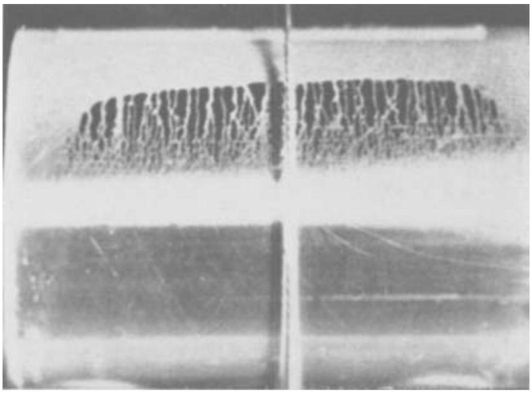

FIGURE 15. Photograph showing the reformation of the oil film for the same value of h_0 as in figure 13 but with $\mu U/T \sim 0{\cdot}004$.

Perspex wedges were cut and placed base downwards between the blocks A and D (figure 5). These are able to give the chamber divergence angles of 1·3° and 2·8°, respectively. They were set with bases at the level of the top of the regulating space so that the maximum values of h_0/h_1 is 1·0, at the upstream end of the observation channel. As U was gradually increased the meniscus became unstable at the same value of $\mu U/T$ as when the channel was parallel and with the same value of h_1 at the open end. The instability which then occurred was very similar to that observed when air is forced into fluid contained between parallel plates (Saffman & Taylor 1958). It developed into air fingers which were separated by fluid columns whose widths were comparable with that of the fingers. Figure 8 (plate 1) is a photograph of the meniscus at that stage using the narrow wedge which made the chamber diverge at 1·3°. The fingers remained in this condition so long as the speed was kept constant (in fact the exposure of this photograph was 8 sec) and their ends remained roughly on a generator of the cylinder corresponding with a constant value of h.

The depth of the observation chamber at the level of the ends of the fingers was measured by inserting strips of flexible plastic material (Polythene) of known thickness as feelers. The results of these measurements are shown in figures 6 and 7 by means of crosses, +, marked A, B, C, D, E in the case of the wider divergence, 2·8°, of the observation chamber, and by squares, ⊡, K, L, M, N for the smaller divergence 1·3°. The points marked A and K represent the values of h_0/h_1 where the meniscus began to retreat from the forward edge of the observation chamber. With the accuracy of the measurements they lie on the curve obtained using the nearly parallel but very slightly *converging* channel obtained by bolting the movable block D (figure 5) directly to the regulating block A without inserting the wedges. The points B and L (figures 6 and 7) represent the points to which the meniscus retracted keeping the speed constant. The photograph of figure 8 (plate 1) corresponds with the point L in figure 6, but as will be seen this point is not very well determined since the ends of the fingers do not all come to rest at the same value of h.

Figures 9 (plate 1) and 10 (plate 2) show the fingers when the divergence was 2·8° and when $\mu U/T$ was 1·4 and 2·2 respectively and correspond with points E and F in figures 6 and 7. Figure 11 shows the fingers when the divergence was 2·8° and $\mu U/T$ was 4·0. This is outside the range of figures 6 and 7 but m was hardly distinguishable from the value it had in the case $\mu U/T = 2{\cdot}2$.

It will also be noticed in figures 8–10 that the fluid separating the air fingers extends to the leading edge of the Perspex block, but on figure 11 (plate 2) these fluid sheets have become so thin that surface tension pulls them back into the wedge-shaped observation chamber so that they part from the rear edge of the stationary block and the saddle-shaped meniscus appears as a bright spot. Similar effects have been noticed by Pearson (1960) and by Floberg (1961 *a, b, c*) using cylinder and plane corresponding with case (*b*) and by Pitts & Greiller (1961) for case (*a*).

It will be noticed that except for the case when the meniscus first leaves the stabilizing edge of the adjustable block the ends of the fingers lie remarkably closely on a line of constant h and that as the speed increases this line, whose movement is represented by the dotted curve in figures 6 and 7, seems to approach more and more closely to the same asymptote as that of the two-dimensional or cylindrical meniscus which is about to leave the stabilizing edge and become unstable. Thus the asymptotic condition for large $\mu U/T$ seems to be one in which the flow separates, so that most of the fluid which leaves the region of separation is carried away on the cylinder in a sheet of uniform thickness, and only a very little is carried in the form of the thin sheets perpendicular to the cylinder which separate the air fingers. This point is referred to later and seems to be verified in figure 15 (plate 4).

10. Cavitation

The separation just described necessarily involves a pressure gradient upstream of the meniscus which may lead to pressures which are so low that true cavitation must occur. Banks & Mill (1954), for instance, using apparatus in which both surfaces moved (case (a)), showed photographs of cavitation bubbles appearing at the point of lowest pressure in the flooded nip between two rotating cylinders. At speeds where the cavitation pressure is just attained at the point of minimum pressure these bubbles will not grow. They may disappear or they may be carried out as small bubbles. When the speed is higher so that the cavitation pressure extends over a larger area the bubbles will grow and will alter the pressure in the fluid round them. This stage of cavitation has been studied by Floberg (1961a). Finally, they may meet and form a continuous air space, or they may extend round the bearing in the form of fingers or streaks which do not join. This kind of cavitation seems to be that visualized by Swift (1932) and Stieber (1933). Curve (a) of figure 2 shows the non-dimensional pressure curve for a flooded nip for which $\lambda = \frac{4}{3}$. Bubbles will appear first at the point A. As the bubble spreads the minimum pressure will be reduced and λ will therefore also be reduced. This process can proceed till the bubbles extend to the atmosphere and the pressure at the level of their vertices is atmospheric. The point B is then reached on the curve $\lambda = 1{\cdot}225$, where $p = 0$ and $dp/dx = 0$. The fingers of air cannot penetrate further because that would involve separation in the part of the pressure curve where dp/dx is positive which is impossible. The point B, figure 2, represents Swift's condition in the case of a very eccentric bearing which is flooded upstream.

The shapes of the air fingers in Swift's type of cavitation must depend on two independent causes both of which tend to make the flooded areas between the fingers get narrower downstream. One cause is geometrical and is the widening of the gap h, and the other is the transfer of fluid beneath the meniscus. The points of the air fingers may be expected to be paraboloidal while they are still narrower than h, since that is the axisymmetrical shape whose cross-section increases linearly. This paraboloidal part would only extend to a length comparable with h and thereafter the shape will be determined by the two causes. If only the first cause were operative, flooded areas between the fingers would occupy a proportion h/h_s of the whole, h_s being the value of h at the points of the fingers. In that case the air spaces would not join together; Cole & Hughes (1956) show some examples of this. The second cause would make the flooded areas get narrower more rapidly than would be expected from purely geometrical considerations.

These speculations resulted from a study of some photographs sent me by Prof. J. A. Cole. They show the cavitation of oil through a loaded transparent bush on a rotating shaft. Figure 12 (plate 3) shows the air fingers (black) when the shaft 0·9820 in. in diameter is rotating at 138 rev/min in a bush 0·9840 in. in diameter and 1·63 in. long. Figure 13 (plate 3) shows the same bearing with the same load, the shaft rotating at 34·6 rev/min. It was not possible to measure h but it is clear that it must have been greater at 138 than at 34 rev. This may account for the fact that the fingers are a greater distance apart in figure 12 than in figure 13. If the geometrical cause for the narrowing of the oil streaks were the only one operating it would have been expected that reduction in their width at a given distance from the beginning of cavitation would be less in figure 12, than in figure 13. The fact that this is not true shows that the second cause, namely the transfer of fluid across the meniscus is operative and is probably the principal cause of the much wider angle of the pointed end of the air finger in figure 12 than in figure 13.

Prof. Cole measured the viscosity of the oil in each of his experiments. He did not measure T but in most oils T lies between 20 and 40 dyn/cm. Using $T = 30$ dyn/cm the values of $\mu U/T$ in the experiments shown in figures 12 and 13 are 0·12 and 0·03, respectively. No measurements of h_0 were made but Prof. Cole calculated the value of the eccentricity ϵ using the theory of Sassenwald & Walther (1954), but interpolating between values calculated by these authors in order to make them applicable to his apparatus. The results were $\epsilon = 0{\cdot}61$ for figure 12 and $\epsilon = 0{\cdot}88$ for figure 13. These correspond with $h_0 = 1{\cdot}0 \times 10^{-3}$ cm for figure 12 and $h_0 = 0{\cdot}3 \times 10^{-3}$ cm for figure 13. The average distance between the fingers in these two cases is 0·22 cm and 0·065 cm so that in each case the spacing of the fingers is about 220 times the minimum clearance distance.

11. Comparison of two types of cavitation

Comparison of figures 12 and 13 with figures 8–11 reveals the physical difference between the two types. In separation cavitation the motion is mainly two-dimensional. The thin partitions between the air fingers carry only a small part of the fluid, the rest is carried in a thin sheet on the moving surface. In true cavitation, starting inside the fluid, much of it is carried in columns filling the space between the two surfaces and separated by air fingers. In bearings these columns may be carried round the shaft unbroken or they may break down leaving a sheet of lubricant of variable thickness on the shaft. When this happens the meniscus which is the boundary of the region where the oil film is re-formed on the far side of the bearing may be expected to reproduce approximately the pattern of the fingers formed in the cavitation region. Figure 14 (plate 4) is one of Prof. Cole's photographs showing the re-formation of the oil film under conditions where widely separated air fingers were formed, probably owing to cavitation of the Swift–Stieber type.

It will be noticed that the air fingers grow wider through their whole length till the oil column between them nearly disappears before the oil film is re-formed. This can only be because the oil is passing under the menisci which form the edges of the oil columns.

Sometimes, however, Prof. Cole obtained quite a different kind of re-formation meniscus. Figure 15 (plate 4) is an example. In that case both the speed and the load were each one-eighth of that used in the experiment of figure 13 so that the eccentricity and, therefore the geometry in the two cases, should be nearly identical. On the other hand, both the maximum negative pressure which would exist if the film were continuous, and the parameter $\mu U/T$ were only one-eighth of that appropriate to figure 13. It is therefore to be expected that true cavitation would be more likely to occur under the conditions of figure 13 than those of figure 15. Unfortunately the illumination failed to show the part of the bearing where cavitation began in figure 15, but the fact that the re-forming meniscus is smooth suggests that the layer of oil carried found on the shaft was of uniform thickness. The thin lines seen in the cavitated area can only represent a small proportion of the oil carried round and may well be the remains of the thin films which separate the air fingers produced by separation rather than by true cavitation.

12. Floberg's experiments

Recently Prof. Birkhoff has called my attention to a number of papers by Floberg (1961*a,b,c*). In the third of these he shows photographs very like Prof. Cole's. In the second he shows photographs of cavitation of extra heavy Vactric lubricating oil when a cylinder 8 cm diameter and 8 cm long was rotated at distances $h_0 = 0{\cdot}01$, $0{\cdot}02$, $0{\cdot}04$, and $0{\cdot}06$ cm below a glass plate. Those taken at $h_0 = 0{\cdot}02$, $0{\cdot}04$, and $0{\cdot}06$ cm were very like figures 8–11 except that they are better photographs, but those at $h_0 = 0{\cdot}01$ cm are quite unlike mine. Possibly the former show separation and the latter cavitation. It is therefore of interest to compare the observed positions of the cavitation with those calculated (*a*) using the Swift–Stieber condition, and (*b*) a separation condition based on my experiment.

The experiment of which Floberg published photographs were carried out at speeds of 25 and 100 rev/min or approximately $U = 10{\cdot}5$ and $41{\cdot}9$ cm/sec so that only two values of $\mu U/T$ occurred. Neither the viscosity nor the surface tension of the oil were recorded, but in answer to a letter, Dr Floberg wrote that at 22 °C the viscosity was 3·5 poise (i.e. g/cm sec). The surface tension was not known but in most lubricating oils it lies in the range 20–40 dyn/cm so that approximate values of $\mu U/T$ can be estimated by assuming $T = 30$. The experiments were carried out under conditions where the oil space was flooded upstream of the nip so that the curves of figure 7 should be applicable, using equation (9*b*) for determining the pressure upstream of the meniscus. With $R = 4{\cdot}0$ cm, $\mu = 3{\cdot}5$ poise, $U = 10{\cdot}5$ and $41{\cdot}9$ cm/sec approximate values of p/p' in dyn/cm² are given in table 2. The approximate values of $\mu U/T$ assuming $T = 30$ dyn/cm are given in the last column of the table.

To find the position at which a two-dimensional meniscus could exist it is theoretically necessary to know δp, the pressure difference between the two sides of the meniscus. This is not known except when $\mu U/T$ is small when it is $2T/h$.

When $\mu U/T$ is large the pressure difference must be of order $\mu U/h$. Since $2T/h$ and $\mu U/h$ are less than $2T/h_0$ and $\mu U/h_0$, it is useful to record the latter, when finding out whether it is justifiable to neglect the effect of δp on the position of the meniscus. The relevant figures are given in the last three lines of table 2. It will be seen that they are of order 1/100 of p/p' in Floberg's experiments.

h_0 (cm)	0·01	0·02	0·04	0·06	$\mu U/T$
p/p' when $U = 10{\cdot}5$	$6{\cdot}2\times 10^5$	$2{\cdot}2\times 10^5$	8×10^4	4×10^4	1·2
p/p' when $U = 41{\cdot}9$	$2{\cdot}5\times 10^6$	$8{\cdot}8\times 10^5$	3×10^5	$1{\cdot}3\times 10^5$	4·9
$2T/h_0$ dyn cm^{-2}	6×10^3	3×10^3	$1{\cdot}5\times 10^3$	10^3	—
$\mu U/h_0$ when $U = 10{\cdot}5$	$3{\cdot}2\times 10^3$	$1{\cdot}6\times 10^3$	8×10^2	5×10^2	1·2
$\mu U/h_0$ when $U = 41{\cdot}9$	$1{\cdot}5\times 10^4$	7×10^3	4×10^3	$2{\cdot}4\times 10^3$	4·9

TABLE 2. Data relating to Floberg's experiments with a cylinder rotating under a transparent flat surface.

In practically all calculations relating to hydrodynamic lubrication it has been assumed that the change in pressure on passing through the meniscus is negligible, so that the pressure distribution when the oil space is flooded upstream depends only on the position of the ends of the cavitation fingers, and is represented in figure 3 by the curve of constant λ which cuts $p' = 0$ at the position of the meniscus.

To estimate the error in the meniscus position due to the neglect of δp assuming that m is a function of $\mu U/T$ only, note that $m = \lambda\cos^2\theta$ so that the error $\delta\theta$ is $(\delta\lambda/2\lambda)\tan\theta$. The change in p' is

$$\left(\frac{\partial p'}{\partial\theta}\right)_\lambda \delta\theta + \left(\frac{\partial p'}{\partial\lambda}\right)_\theta \delta\lambda = \delta\theta\left[\left(\frac{\partial p'}{\partial\theta}\right)_\lambda + \left(\frac{\partial p'}{\partial\lambda}\right)_\theta (2\lambda\cot\theta)\right].$$

The values of $(\partial p'/\partial\theta)_\lambda$ and $(\partial p'/\partial\lambda)_\theta$ can be estimated using figure 3 for any position on $p' = 0$. Thus near $\theta = 45°$, $\lambda = 1{\cdot}27$ and $(\partial p'/\partial\lambda)_\theta$ is approximately 1·0 while $(\partial p'/\partial\theta)_\lambda$ is approximately 0·2, so that $\delta\theta = \delta p'/2{\cdot}8$ and since $\delta p = (p/p')\,\delta p'$, it follows that $\delta\theta = (p'/p)\,\delta p/2{\cdot}8$.

Comparing the figures in the last two lines in table 2 with those in lines 2 and 3 it will be seen that $p'\delta p/p$ is of order 1/200 for $h_0 = 0{\cdot}01$ and about 1/50 for $h_0 = 0{\cdot}06$. The error therefore in neglecting δp and using the relationship between m and θ when $p' = 0$ is only a fraction of a degree in θ. This relationship is given by setting $p' = 0$ in (11). It is

$$m = \lambda\cos^2\theta = \frac{\theta + \frac{1}{2}\pi + \frac{1}{2}\sin 2\theta}{\frac{3}{4}(\theta + \frac{1}{2}\pi) + \frac{1}{2}\sin 2\theta + \frac{1}{16}\sin 4\theta} \tag{23}$$

and it is shown graphically in figure 16.

If the value of m at which a two-dimensional or cylindrical meniscus can exist is a function of $\mu U/T$ only, as it appears to be in my experiments, it is possible to use the experimental curve of figure 7 with the theoretical curve of figure 16 to predict where such a meniscus could exist in Floberg's apparatus. For this purpose figure 17 was constructed showing the relationship between $\mu U/T$ and

$\tan\theta = x(2Rh_0)^{-\frac{1}{2}}$. Curve (1) shows the relationship so found. In my experiments the two-dimensional meniscus was limited to the range $0 < \mu U/T < 1{\cdot}75$. On the other hand, measurements of m corresponding with the ends of the air fingers were carried to $\mu U/T = 2{\cdot}8$ and these are represented in figure 7 by a broken line. This broken line has been transferred to figure 17 as curve (2) by

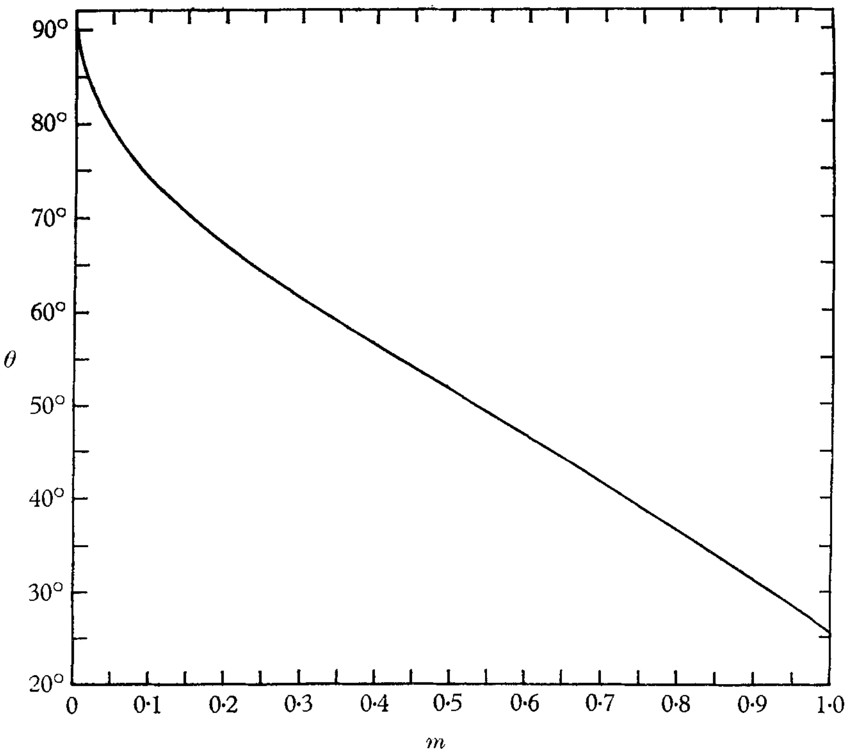

FIGURE 16. Relationship between m and θ for points in figure 3 on $p' = 0$.

the same method as that used to produce curve (1) for the two-dimensional meniscus experiments, but it must be remembered that there is no reason to believe that the broken curve of figure 7 would represent experiments with other distributions of thickness in the narrow oil passage. Though the two-dimensional meniscus was expected *a priori* to depend only on the local conditions near the meniscus the stability of the meniscus certainly depends on the distribution of thickness of the oil passage (Pearson 1960; Greiller & Pitts 1961) and the positions of the ends of the fingers may do so also.

13. Comparison with measurements of Floberg's photographs

The main difficulty in attempting to measure x, the distance of the ends of the air fingers in Floberg's photograph from the narrowest point in the oil passage is that the position of this point was not marked on the photographs. In a letter, however, Dr Floberg mentions that the method of lighting was such that the point $x = 0$ was close to the edge of a dark band which appeared in all his photographs. Using this rough method for locating $x = 0$ the measurements given in column 3 of table 3 were made, and $\tan\theta$ and θ were then tabulated in columns 4 and 5. The corresponding values of $\mu U/T$ are given in column 6 and the points marked on figure 17 with the symbols of column 7, table 3.

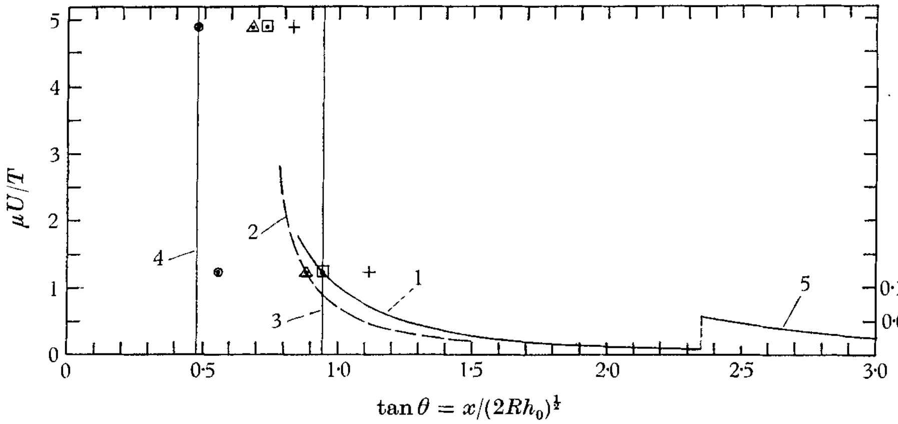

FIGURE 17. Points represent observed positions of the ends of air fingers in Floberg's experiments: △ $h_0 = 0·06$, ⊡ $h_0 = 0·04$, + $h_0 = 0·02$, ⊙ $h_0 = 0·01$. Lines represent: (1) position of two-dimensional meniscus using experimental data of figure 7; (2) positions of ends of fingers using data of figure 7; (3) Hopkins criterion ($m = \frac{2}{3}$ for case (b)); (4) Swift–Stieber criterion; (5) continuation of line (1) when ordinates are increased in ratio 10 : 1.

1 Floberg's plate number	2 h_0 (cm)	3 x (cm)	4 $\tan\theta$	5 θ	6 $\frac{\mu U}{T}$	7 Symbol
30·1	0·06	0·618	0·892	41°·7	1·2	△
30·2	0·06	0·469	0·677	34°·1	4·9	△
31·1	0·04	0·538	0·9515	43°·6	1·2	⊡
31·2	0·04	0·412	0·728	36°·1	4·9	⊡
32·1	0·02	0·45	1·12	48°·2	1·2	+
32·2	0·02	0·33	0·830	39°·7	4·9	+
33·1	0·01	0·160	0·566	29°·50	1·2	⊙
33·2	0·01	0·137	0·485	25°·9	4·9	⊙

TABLE 3. Measurements from Floberg's photographs and corresponding values of $\tan\theta = x/(2Rh_0)^{\frac{1}{2}}$.

It will be seen in figure 17 that the measurements made at $\mu U/T = 1·2$ are near the lines (1) and (2) for $h_0 = 0·02$, 0·04 and 0·06 cm and that they are far from the Swift–Stieber line. On the other hand the point corresponding with $h_0 = 0·01$ cm is close to this line.

Since the highest values of $\mu U/T$ at which I measured the positions of the fingers was 2·8, it is not possible to make a comparison with Floberg's results at $\mu U/T = 4·9$ but the positions of the points representing his photographs is consistent with a gradual increase in m for $h_0 = 0·02$, 0·04 and 0·06 cm as $\mu U/T$ increases. The point for $h_0 = 0·01$ is very close to the Swift–Stieber line when $\mu U/T = 4·9$.

These results seem to show that Floberg in his experiments got both types of cavitation, though he does not distinguish between them. He noticed that his results for $h_0 = 0{\cdot}01$ cm were near the Swift–Stieber point and that the meniscus retreats slightly from the point $x = 0$ as the speed is reduced. This can be seen in figure 17 where the point for $\mu U/T = 1{\cdot}2$, $h_0 = 0{\cdot}01$ cm corresponds with a slightly greater value of x than that for $\mu U/T = 4{\cdot}7$. Floberg does not comment on the fact that in his experiments at $h_0 = 0{\cdot}02$, $0{\cdot}04$ and $0{\cdot}06$ cm, x is very much greater than the Swift–Stieber criterion would allow. One interesting feature of figure 17 is that at each of the values of $\mu T/U$ a *decrease* in h_0 from $0{\cdot}06$ to $0{\cdot}02$ cm *increases* $x/(2Rh_0)^{\frac{1}{2}}$ continuously but a further *decrease* in h_0 to $0{\cdot}01$ cm causes the meniscus to *retreat* to the Swift–Stieber position. This again is evidence that a change in the physical nature of the cavitation took place between $h_0 = 0{\cdot}02$ and $0{\cdot}01$ cm.

14. Possible mode of transformation of one type of cavitation to the other

It is known that internal cavitation occurs in oils at pressures far exceeding their vapour pressure owing to the existence of gases dissolved in them which are released with a comparatively small drop in pressure. This probably accounts for the fact that the greatest decrease in pressure calculated even for a flooded bearing, Floberg's experiment at $h_0 = 0{\cdot}01$ cm and $U = 10{\cdot}5$ cm/sec, is far less than an atmosphere, yet the Swift–Stieber condition seems to apply fairly closely.

Table 2 shows that the value $-p/p'$ was greater for $\mu U/T = 4{\cdot}9$, $h_0 = 0{\cdot}02$ cm than for $\mu U/T = 1{\cdot}2$, $h_0 = 0{\cdot}01$ cm, but figure 17 suggests that the Swift–Stieber cavitation did not occur in the former case but did in the latter. This may well be due to the fact that separation cavitation necessarily implies a reduction in pressure upstream of the meniscus. If this is sufficiently great internal cavitation may take place, but will be seen from figure 3 that as the position of the separation meniscus moves back along the axis $p' = 0$ the maximum pressure defect, $-p'$, decreases. Thus though p/p' is greater than $\mu U/T = 4{\cdot}9$, $h_0 = 0{\cdot}2$ cm than for $\mu U/T = 1{\cdot}2$, $h_0 = 0{\cdot}1$ cm the pressure defect at the pressure minimum is probably less in the former case than the latter.

In conclusion I wish to express my thanks to Prof. J. A. Cole for his photographs (figures 12–15), to Dr Leif Floberg for information about his experiments, and to Prof. Garrett Birkhoff for some useful discussion.

Appendix

Speculations on the uniqueness of magnetical solutions of flow problems involving free surfaces

The mathematical problem which would have to be solved to represent the flow when a viscous fluid is driven from a tube by air pressure applied at one end is very difficult even when only two-dimensional flow is considered. In that case the problem is to represent the meniscus between a viscous fluid and air when one or both of two bounding parallel planes move relative to it. The flow can be represented by a stream function ψ satisfying the field equation $\nabla^4\psi = 0$ and the flow can be made steady by an appropriate translation of the whole field. The boundary conditions to be satisfied at the planes are that the two components of velocity are the same as those of the planes. On the meniscus, however, three conditions must be satisfied. These are:

(i) the component of velocity normal to the meniscus is zero;

(ii) the component of shear stress parallel to its surface is zero;

(iii) the component of stress normal to its surface is (surface tension)/(radius of curvature of the meniscus).

Since only two of these conditions can be satisfied at an arbitrarily chosen surface the possibility of satisfying the third can only be realized by varying the shape of the meniscus. An experimenter would certainly expect a definite shaped meniscus to establish itself, but it may be very difficult to prove uniqueness mathematically.

Some light might be thrown on this subject by recalling some simpler problems relating to the flow of an ideal non-viscous fluid with a free surface. Here the flow is irrotational and at fixed boundaries only one condition can be satisfied, but at a free surface it is possible to satisfy two. For instance, many free-surface problems involving steady flow under the action of gravity have been solved analytically, or by numerical processes and it has been recognized that the solutions are often not unique. The stationary waves produced by an obstacle on the bed of a stream, for instance, is perhaps a trivial example. In that case solutions exist corresponding with cases where waves of arbitrary amplitude and phase are propagated upstream of the obstacle at the speed of the stream. There is a unique case where no such waves exist and it has been pointed out that even the smallest viscosity would tend to make this case the nearest approximation to the real phenomenon exhibited by an obstacle in the bed of a smoothly flowing stream.

In the case just cited the field extends to infinity but cases of non-unique flow with a closed free surface can be imagined. Consider, for instance, a free vortex partially filling a rigid circular cylindrical case. In one example the free surface might be a concentric cylinder, but capillary ripples can exist on the free surface and if these are of such a length that they can travel backwards at the speed of the fluid at the free surface the motion is an alternative steady flow satisfying all the necessary boundary conditions. This example is peculiar because for a given amount of fluid in the rigid cylindrical boundary circular free steamlines are possible with all vortex strengths, but the alternative steady motions are only possible for discrete values of this strength, namely for those for which the circular free streamline is an integral number of the critical wavelength which can travel backwards at the speed which makes steady motion possible.

The non-uniqueness of solutions where waves can exist is well understood but recently a more interesting case has been found. Garabedian (1957) has shown that there is a single infinite set of symmetrical solutions of the two-dimensional version of the problem of emptying water from a vertical tube which is closed at the top. Here both intuition, and experimentation in the axisymmetric case, lead to the expectation that only one of the solutions would represent the actual phenomenon but there seems to be no convincing reason for the choice of any particular solution. Garabedian pointed out that one of them represents the case where the air column rises at a maximum rate, but there seems little justification for choosing that particular solution as the one which would occur if the two-dimensional bubble could be realized experimentally. (It has been pointed out to me by Garrett Birkhoff in a private communication that though one of Garabedian's solutions represents the symmetrical bubble which rises at maximum speed a larger asymmetrical bubble which would rise at a speed $\sqrt{2}$ times Garabedian's maximum is theoretically possible.)

Another case where an infinity of solutions of $\nabla^2\psi = 0$ can satisfy the relevant boundary conditions is provided by a Hele–Shaw cell (Saffman & Taylor 1958). In that case it was shown experimentally that only one of the motions described by these solutions can be set up. Recently Jacquard & Séquier (1962) have shown, by tracing theoretically a method of setting up the motion, that the experimentally observed motion would result from the mode of imitation which they analyse.

Uniqueness of flow when viscous fluid is blown from a tube

Acceptance of equation (1) pre-supposes uniqueness of flow. This is in accordance with experimental measurements made with tubes of varying bore and viscosity. Theoretical justification for (1) based solely on dimensional arguments is not convincing. Dimensional arguments can justifiably be used to state that if m is known all cases of flow are similar for a given value of $\mu U/T$. The justification for taking m as a function of $\mu U/T$ must be based either on detailed analysis of the flow, which at present seems to be outside the range of practical possibility for mathematicians, or on physical intuition. When the tube is flooded upstream from the meniscus (upstream when brought to rest by appropriate translation), as it is when fluid is blown from the tube by pressure applied at one end, μ, T, the radius of the tube a, and the pressure gradient in the flooded portion, dp/dx, are the parameters which can be varied at will. Poiseuille's equation connects $U/(1-m)$, μ and a with dp/dx so that we can regard μ, T, a and $U/(1-m)$ as the factors variable at will. The only non-dimensional combination of these is $\mu U/T(1-m)$, but there is no simple way in which a one-to-one relationship between $\mu U/T$ and $1-m$ can be established.

When the U is reversed so that fluid is blown into a tube which already contains fluid distributed uniformly on the walls, $\mu U/T(1-m)$ is still variable at will but since m is now fixed any value of $\mu U/T$ can occur for any value of m.

It is curious that crude dimensional theory can apply when U is positive but not when it is negative. It seems reasonable therefore to expect that a mathematical solution when found will not be unique, but that the physical situation will be unique when U is positive and $\mu U/T$ is known, whereas when U is negative it is necessary to know m as well as $\mu U/T$ in order to establish a physical situation which is similar for variations in μ, U, T and a.

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